View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

Indag. Mathem., N.S., 4 (2), 129-140 June 21, 1993

Ol'shanskiĭ wedges in symmetric Lie algebras

by Norbert Dörr*

Technische Hochschule Darmstadt, D-6100 Darmstadt, Germany

Communicated by Prof. W.T. van Est at the meeting of June 22, 1992

ABSTRACT

Ol'shanskiĭ's semigroup plays a prominent role in the discussion of symmetric spaces. There are two important applications: the construction of the discrete series in representation theory and analysis on symmetric spaces. In this article the notion of an Ol'shanskiĭ wedge in a symmetric Lie algebra is defined. The tangent wedge of Ol'shanskiĭ's semigroup is an example of such a wedge. In Section 1 of this paper, a geometric characterization of Ol'shanskiĭ wedges is given and their relation to special Lie wedges is established. Section 2 deals with invariant Ol'shanskiĭ wedges and Ol'shanskiĭ semialgebras. We give a complete classification of symmetric Lie algebras supporting invariant Ol'shanskiĭ wedges, resp., Ol'shanskiĭ semialgebras.

INTRODUCTION

In all that follows, let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ be a finite dimensional real *symmetric* Lie algebra. Then $\mathfrak{h}$ is a subalgebra, and the vector subspace $\mathfrak{q}$ is an $\mathfrak{h}$-module with $[\mathfrak{q},\mathfrak{q}]\subseteq\mathfrak{h}$. The decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ determines an involutive Lie algebra automorphism $\tau:\mathfrak{g}\to\mathfrak{g}$ with eigenspaces $\mathfrak{h}$ and $\mathfrak{q}$ for the eigenvalues $+1$ and -1, and vice versa. Symmetric Lie algebras play a prominent role in the discussion of *symmetric spaces* G/H, where G is a connected Lie group with involutive Lie group automorphism $\hat{\tau}$ with $d\hat{\tau}=\tau$, and $(G^{\hat{\tau}})_0\subseteq H\subseteq G^{\hat{\tau}}$, the set of fixed points in the involution $\hat{\tau}$.

In this article we consider wedges W in $\mathfrak{g}$, where it is assumed that $\mathfrak{h}$ is contained in the edge $H(W)=W\cap -W$ of W. These are precisely the wedges of the

* The author thanks Prof. Dr. Karl H. Hofmann for his support.

form $W = \mathfrak{h} \oplus C$ with a wedge C in $\mathfrak{q}$. Wedges of this type occur as tangent objects of important semigroups in the associated simply connected Lie group G, which give rise to a structure of an order on the symmetric space G/H where H is the analytic subgroup with Lie algebra $\mathfrak{h}$. The order is a partial order, if C is pointed. In [1], this type of semigroups is discussed in the case where $\mathfrak{g}$ is the complexification of a real Lie algebra, viewed as a real Lie algebra. One important application in representation theory is the construction of the holomorphic discrete series, which is described by Ol'shanskiĭ in [6]. Ol'shanskiĭ's semigroup is also a useful tool in analysis on symmetric spaces. Here the work of Hilgert, 'Olafsson and Ørsted [3] should be mentioned.

It is assumed that the reader is familiar with wedges in Lie algebras. A detailed discussion of this topic may be found in [2]. We recall that a wedge W in a Lie algebra $\mathfrak{g}$ is called a *Lie wedge*, resp., an *invariant wedge* if $e^{\mathrm{ad}\, H(W)} W = W$, resp., $e^{\mathrm{ad}\, \mathfrak{g}} W = W$. This is the case if and only if $[H(W), x] \subseteq T_x(W)$, resp., $[\mathfrak{g}, x] \subseteq T_x(W)$ for all $x \in W$, where $T_x(W)$ is the tangent space to the wedge W at x. We denote by $*$ the Campbell-Hausdorff multiplication. A wedge W is called a *Lie semialgebra* if there exists a Campbell-Hausdorff-neighborhood U in $\mathfrak{g}$ such that $(W \cap U) * (W \cap U) \subseteq W$. This is equivalent to $[T_x(W), x] \subseteq T_x(W)$ for all $x \in W$.

In Section 1 we define the notion of *Ol'shanskiĭ wedges*. Consider a wedge $W = \mathfrak{h} \oplus C$ in a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Then C should be invariant under inner automorphisms which come from the subalgebra $\mathfrak{h}$, that is, we want $e^{\mathrm{ad}\, \mathfrak{h}} C = C$. The invariance under $\mathfrak{h}$ enables us to define a causal structure on the symmetric space G/H, i.e. to each point $x \in G/H$ we assign a closed convex cone $\theta(x)$ in the tangent space $T_x(G/H)$, such that the action of G on G/H preserves the cone field. We also demand that an invariance condition similar to the Lie semialgebra condition is satisfied. We find a characterization in terms of the tangent spaces, and prove that Ol'shanskiĭ wedges are exactly those Lie wedges which contain $\mathfrak{h}$ in their edge.

In Section 2 we consider further invariance properties. We ask which properties the wedge W and the symmetric Lie algebra $\mathfrak{g}$ must have, if we require W to be a Lie semialgebra or, to be invariant even. We give a complete classification of symmetric Lie algebras supporting *invariant* Ol'shanskiĭ wedges, resp., Ol'shanskiĭ *semialgebras* $W = \mathfrak{h} \oplus C$ where C is pointed and generating.

1. CHARACTERIZATION OF OL'SHANSKIĬ WEDGES

For a wedge W in a Lie algebra $\mathfrak{g}$ the subtangent wedge $L_x(W)$ at $x \in W$ is defined by $L_x(W) = \overline{W - \mathbb{R}^- \cdot x}$. Its edge $T_x(W) = L_x(W) \cap -L_x(W)$ is just the tangent wedge of W at x.

DEFINITION 1. A wedge W in a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is an *Ol'shanskiĭ wedge*, if there is a wedge $C \subseteq \mathfrak{q}$ such that $W = \mathfrak{h} \oplus C$ and the following holds:

(1) $e^{\mathrm{ad}\, \mathfrak{h}} C = C$.

(2) There is a Campbell-Hausdorff-neighborhood B in $\mathfrak{g}$ such that

$$g(\operatorname{ad} x)L_x(W) = L_x(W)$$

for all $x \in C \cap B$.

Condition (1) implies that $C - C$ and $H(C)$ are $\mathfrak{h}$-modules. Obviously, $W - W = \mathfrak{h} \oplus (C - C)$ is a symmetric subalgebra of $\mathfrak{g}$ in which W is generating. Therefore, we often need only to consider *generating* Ol'shanskiĭ wedges.

PROPOSITION 1.2. *Let $W = \mathfrak{h} \oplus C$ be an Ol'shanskiĭ wedge in a finite dimensional symmetric Lie algebra and $w = h + c \in W$. Then*

$$L_w(W) = \mathfrak{h} \oplus L_c(C) \quad \textit{and} \quad T_w(W) = \mathfrak{h} \oplus T_c(C).$$

Further, if W is generating, then $w \in C^1(W)$ if and only if $c \in C^1(C)$.

The following lemma gives a characterization of Lie wedges W in a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ with $\mathfrak{h} \subseteq H(W)$.

LEMMA 1.3. *Let $W = \mathfrak{h} \oplus C$ be a wedge in $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Then the following conditions are equivalent*:

(1) *W is a Lie wedge.*
(2) $e^{\operatorname{ad} \mathfrak{h}} C = C$.
(3) $[c, \mathfrak{h}] \subseteq T_c(C)$ *for all* $c \in C$.

PROOF. (2) $\Leftrightarrow$ (3) This is a consequence of the Linear Invariance Theorem for Wedges and Vector Fields [2] I.5.23.

(1) $\Rightarrow$ (2) W is a Lie wedge if and only if $e^{\operatorname{ad} H(W)} W = W$. In particular, (1) implies that $e^{\operatorname{ad} \mathfrak{h}} W = W$ holds. But $e^{\operatorname{ad} \mathfrak{h}} \mathfrak{h} = \mathfrak{h}$ and $e^{\operatorname{ad} \mathfrak{h}} \mathfrak{q} = \mathfrak{q}$ imply $e^{\operatorname{ad} \mathfrak{h}} C = e^{\operatorname{ad} \mathfrak{h}}(W \cap \mathfrak{q}) = W \cap \mathfrak{q} = C$.

(3) $\Rightarrow$ (1) The edge $H(C)$ is an $\mathfrak{h}$-module, and $T_w(W) = \mathfrak{h} \oplus T_c(C)$ for $w = h + c \in W$. Further $[H(C), c] \subseteq [\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}$. Thus $[H(W), w] = [\mathfrak{h} + H(C), h + c] \subseteq [\mathfrak{h}, h] + [H(C), h] + [\mathfrak{h}, c] + [H(C), c] \subseteq \mathfrak{h} + T_c(C) = T_w(W)$. But this is equivalent to W being a Lie wedge. □

PROPOSITION 1.4. *Suppose that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is a symmetric Lie algebra with a wedge $W = \mathfrak{h} \oplus C$ and suppose further that B is a $C - H$-neighborhood such that* $\operatorname{Spec}(\operatorname{ad} x) \cap 2\pi i \cdot \mathbb{Z} = \{0\}$ *for all $x \in B$. Then the following conditions are equivalent*:

(1) $e^{\operatorname{ad} \mathfrak{h}} C \subseteq C$.
(2) $[c, T_c(W)] \subseteq T_c(W)$ *for all* $c \in C$.
(3) $g(\operatorname{ad} c) W \subseteq L_c(W)$ *for all* $c \in C \cap B$.
(4) $g(\operatorname{ad} c) L_c(W) \subseteq L_c(W)$ *for all* $c \in C \cap B$.
(5) $g(\operatorname{ad} c) T_c(W) = T_c(W)$ *for all* $c \in C \cap B$.

And, if C is generating,

(6) $[c, \mathfrak{g}] \subseteq T_c(W)$ *for all* $c \in C$.

If C is generating, we may take $c \in C^1(C)$ in (1)–(6).

PROOF. (1) $\Rightarrow$ (2) By Lemma 1.3 we have $[c,\mathfrak{h}] \subseteq T_c(C)$ for all $c \in C$. By Proposition 1.2, $T_c(W) = \mathfrak{h} \oplus T_c(C)$. Then we have $[c, T_c(W)] = [c,\mathfrak{h}] + [c, T_c(C)] \subseteq T_c(C) + \mathfrak{h} = T_c(W)$. Therefore (2) holds.

(2) $\Rightarrow$ (1) We have, in particular, $[c,\mathfrak{h}] \subseteq T_c(W)$. Let $w = h + c \in W$. Then $[w,\mathfrak{h}] = [h+c,\mathfrak{h}] \subseteq [h,\mathfrak{h}] + [c,\mathfrak{h}] \subseteq \mathfrak{h} \oplus T_c(W) = T_w(W)$. This is equivalent to $e^{\operatorname{ad}\mathfrak{h}} W = W$. Again, this implies $e^{\operatorname{ad}\mathfrak{h}} C = C$, i.e., (1).

(2) $\Leftrightarrow$ (5) This follows from [2] II.2.12.

(3) $\Rightarrow$ (4) Let $c \in C \cap B$. Then

$$g(\operatorname{ad} c) L_c(W) \subseteq \overline{g(\operatorname{ad} c)(W - \mathbb{R}^+ \cdot c)} \subseteq \overline{g(\operatorname{ad} c) W - \mathbb{R}^+ \cdot c} \subseteq L_c(W).$$

This shows (4).

(4) $\Rightarrow$ (5) This is obvious since $g(\operatorname{ad} c)$ is a linear map and $T_c(W) = L_c(W) \cap -L_c(W)$.

(5) $\Rightarrow$ (3) Since (5) is equivalent to (1), we have in particular that $C - C$ is an $\mathfrak{h}$-module and therefore $W - W$ is a Lie algebra. Thus we may assume that W is generating. By Proposition 1.2, $C^1(W) \cap C = C^1(C)$. Let $c \in C^1(C)$. Then $L_c(W)$ is a halfspace which is boundec by $T_c(W)$. We have $g(t \cdot \operatorname{ad} c) T_{t \cdot c}(W) \subseteq T_{t \cdot c}(W)$ for all $c \in C^1(C) \cap B$ and $t \in [0,1]$. Since $T_{t \cdot c}(W) = T_c(W)$ we have $g(t \cdot \operatorname{ad} c) T_c(W) \subseteq T_c(W)$ for all $t \in [0,1]$. Therefore $g(t \cdot \operatorname{ad} c)$ leaves the two halfspaces bounded by $T_c(W)$ invariant or exchanges them. Take any $w \in \operatorname{int}(C)$. Then $t \mapsto g(t \cdot \operatorname{ad} c) w$ is a continuous curve, which does not intersect $T_c(W)$, since $g(t \cdot \operatorname{ad} c)$ is bijective and leaves $T_c(W)$ invariant. Therefore the halfspace bounded by $T_c(W)$ which contains C, that is $L_c(W)$, is $g(\operatorname{ad} c)$-invariant. This shows (3).

If C is generating then the equivalence of (2)–(5) and the corresponding conditions with $c \in C^1(C)$ follow by the Invariance Theorem for Wedges and Vector Fields [2] I.5.23.

(2) $\Leftrightarrow$ (6) If (2) holds, we have in particular $[c,\mathfrak{h}] \subseteq T_c(W)$. Let $w = h + c' \in W$. Then $[c, w] = [c, h] + [c, c'] \in T_c(W) + [\mathfrak{q}, \mathfrak{q}] \subseteq T_c(W)$. Thus $[c, W - W] \subseteq T_c(W)$. Since W is generating this implies (6). That (2) follows by (6) is trivial. $\square$

Summarizing the results of Lemma 1.3 and Proposition 1.4 we get the following characterization of Ol'shanskiĭ wedges.

THEOREM 1.5. (Characterization Theorem for Ol'shanskiĭ Wedges) *For a wedge $W = \mathfrak{h} \oplus C$ in a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ the following conditions are equivalent:*

(1) *W is an Ol'shanskiĭ wedge.*

(2) *W is a Lie wedge.*

(3) $e^{\operatorname{ad}\mathfrak{h}} C = C$.

(4) $[\mathfrak{h}, c] \subseteq T_c(C)$ *for all* $c \in C$ (*resp.*, $c \in C^1(C)$, *if C is generating*).

2. INVARIANT OL'SHANSKIĬ WEDGES AND OL'SHANSKIĬ SEMIALGEBRAS

In the preceding section we have seen that the notion of Ol'shanskiĭ wedges in symmetric Lie agebras is entirely equivalent to Lie wedges which contain the subalgebra $\mathfrak{h}$ in the edge.

DEFINITION 2.1. An Ol'shanskiĭ wedge W in a symmetric Lie algebra $\mathfrak{g}$ is an *Ol'shanskiĭ semialgebra*, resp., an *invariant Ol'shanskiĭ wedge* if and only if W is a Lie semialgebra, resp., an invariant wedge.

PROPOSITION 2.2. *For a wedge $W=\mathfrak{h}\oplus C$ in a symmetric Lie algebra $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ the following conditions are equivalent*:

(1) $T_x(W)$ *is an ideal for all* $x\in W$.

(2) W *is an invariant Ol'shanskiĭ wedge.*

(3) $[\mathfrak{q},\mathfrak{h}]\subseteq H(C)$.

(4) $H(W)$ *is an ideal in* $\mathfrak{g}$.

If C is pointed, then these conditions are equivalent to

(3′) $[\mathfrak{q},\mathfrak{h}]=\{0\}$.

(4′) $\mathfrak{h}$ *is an ideal of* $\mathfrak{g}$.

PROOF. First, we remark that in any case $C-C$ is an $\mathfrak{h}$-module. Therefore we may assume C to be generating. Then for every $x=h+c\in W$ we have $T_x(W)=\mathfrak{h}\oplus T_c(C)$.

(3) $\Leftrightarrow$ (4) This is straightforward since $H(W)=\mathfrak{h}\oplus H(C)$.

(1) $\Rightarrow$ (2) If $T_x(W)$ is an ideal, then in particular $[x,\mathfrak{g}]\subseteq T_x(W)$ since $x\in T_x(W)$. But this is equivalent for W to be invariant.

(2) $\Rightarrow$ (3) Let W be an invariant Ol'shanskiĭ wedge and $c\in C$. For any $h\in\mathfrak{h}$ we have $[\mathfrak{q},h+c]\subseteq[\mathfrak{g},h+c]\subseteq T_{h+c}(W)=\mathfrak{h}+T_c(C)$. Thus $[\mathfrak{q},c]\subseteq\mathfrak{h}$ and $[\mathfrak{q},h]\subseteq T_c(C)$ for all $h\in\mathfrak{h}$ and all $c\in C$. Thus by [2] I.3.12, $[\mathfrak{q},\mathfrak{h}]\subseteq\bigcap_{c\in C}T_c(C)=H(C)$.

(3) $\Rightarrow$ (1) We have $T_x(W)=\mathfrak{h}+T_c(C)$, $x=h+c$. Then $[T_x(W),\mathfrak{g}]=[\mathfrak{h}+T_c(C),\mathfrak{h}+\mathfrak{q}]\subseteq[\mathfrak{h},\mathfrak{h}]+[T_c(C),\mathfrak{q}]+[T_c(C),\mathfrak{h}]+[\mathfrak{h},\mathfrak{q}]\subseteq\mathfrak{h}+H(C)\subseteq T_x(W)$ since $[\mathfrak{q},\mathfrak{h}]\subseteq H(C)$. This shows (1).

If C is pointed, then $H(C)=\{0\}$. This proves the assertion. □

By Proposition 2.2, symmetric Lie algebras which support invariant Ol'shanskiĭ wedges are of restricted type.

THEOREM 2.3. (Classification of symmetric Lie algebras with invariant Ol'shanskiĭ wedges) *Let $\mathfrak{q}$ be a vector space and $\mathfrak{h}$ a Lie algebra with centre $\mathfrak{z}(\mathfrak{h})$. Further, let $\kappa:\mathfrak{q}\times\mathfrak{q}\to\mathfrak{z}(\mathfrak{h})$ be a skew-symmetric, bilinear map. Then $\mathfrak{g}_\kappa\stackrel{\text{def}}{=}\mathfrak{h}\oplus\mathfrak{q}$ with the bracket*

$$[(h,q),(h',q')]=(\kappa(q,q')+[h,h'],0)$$

for $q,q'\in\mathfrak{q}$, $h,h'\in\mathfrak{h}$ is a Lie algebra for which the following holds:

(1) $\mathfrak{h}$ *is an ideal in* $\mathfrak{g}_\kappa$.

(2) $[\mathfrak{q},\mathfrak{q}]\subseteq\mathfrak{z}(\mathfrak{h})$.

(3) *If $\mathfrak{r}_\mathfrak{h}$ is the radical of $\mathfrak{h}$, then $\mathfrak{r}\stackrel{\text{def}}{=}\mathfrak{r}_\mathfrak{h}\oplus\mathfrak{q}$ is the radical of $\mathfrak{g}_\kappa$, and if $\mathfrak{s}$ is any Levi complement of $\mathfrak{r}_\mathfrak{h}$ in $\mathfrak{h}$, then $\mathfrak{s}$ is also a Levi complement of $\mathfrak{r}$ in $\mathfrak{g}_\kappa$.*

For any pointed *wedge* $C \subseteq \mathfrak{q}$ *the wedge* $W = \mathfrak{h} \oplus C$ *is an invariant Ol'shanskiĭ wedge in* $\mathfrak{g}_\kappa$.

Conversely, if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *is a symmetric Lie algebra supporting an* invariant *Ol'shanskiĭ wedge* $W = \mathfrak{h} \oplus C$ *with* C *pointed, then* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{g}_\kappa$ *with* $\kappa(q, q') = [q, q']$ *for* $q, q' \in \mathfrak{q}$.

PROOF. We show that $\mathfrak{g}_\kappa$ is a Lie algebra. Bilinearity and skew-symmetry of $[\cdot, \cdot]$ are obvious. Let $q, q', q'' \in \mathfrak{q}$ and $h, h', h'' \in \mathfrak{h}$. Then $[(h, q), [(h', q'), (h'', q'')]] = [(\kappa(q', q'') + [h', h''], 0), (h, q)] = ([h, [h', h'']], 0)$, since $\kappa(q', q'') \in \mathfrak{z}(\mathfrak{h})$. Thus the Jacobi identity holds.

The assertions (1) and (2) follow from the definitions. let $\mathfrak{s}$ be a Levi complement for $\mathfrak{r}_\mathfrak{h}$ in $\mathfrak{h}$. First,

$$\begin{aligned}
[\mathfrak{r}, \mathfrak{g}_\kappa] &\subseteq [\mathfrak{r}_\mathfrak{h} + \mathfrak{q}, \mathfrak{h} + \mathfrak{q}] \\
&\subseteq \mathfrak{r}_\mathfrak{h} + [\mathfrak{q}, \mathfrak{h}] + [\mathfrak{r}_\mathfrak{h}, \mathfrak{q}] + [\mathfrak{q}, \mathfrak{q}] \\
&\subseteq \mathfrak{r}_\mathfrak{h} + \{0\} + \{0\} + \mathfrak{z}(\mathfrak{h}) \\
&\subseteq \mathfrak{r}_\mathfrak{h}
\end{aligned}$$

implies that $\mathfrak{r}$ is a solvable ideal with $\mathfrak{g}_\kappa = \mathfrak{r} \oplus \mathfrak{s}$. Also $\mathfrak{s} \cong \mathfrak{g}_\kappa / \mathfrak{r} \cong \mathfrak{h}/\mathfrak{r}$ is a semisimple subalgebra complementary to $\mathfrak{r}$. But this implies (3).

Now, let C be a pointed cone in $\mathfrak{q}$. Since $[\mathfrak{q}, \mathfrak{h}] = \{0\}$, the wedge $W = \mathfrak{h} \oplus C$ is an invariant Ol'shanskiĭ wedge due to Proposition 2.2.

Conversely, let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra supporting an invariant Ol'shanskiĭ wedge $W = \mathfrak{h} \oplus C$ with C pointed. By Proposition 2.2, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ and $\mathfrak{q}$ is contained in the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. In particular we have $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{z}(\mathfrak{h})$ using the Jacobi identity. Define $\kappa : \mathfrak{q} \times \mathfrak{q} \to \mathfrak{z}(\mathfrak{h})$, $\kappa(q, q') = [q, q']$ and g_κ as above. Then g_κ is isomorphic to $\mathfrak{g}$. This proves the assertion. □

Recall, a wedge W is called a *Lie semialgebra* if there exists a Campbell–Hausdorff-neighborhood U in $\mathfrak{g}$ such that $(W \cap U) * (W \cap U) \subseteq W$. This is equivalent to $[T_x(W), x] \subseteq T_x(W)$ for all $x \in W$. An Ol'shanskiĭ wedge which has this property is called an Ol'shanskiĭ semialgebra. Our aim is also to give a complete classification of symmetric Lie algebras supporting Ol'shanskiĭ semialgebras $W = \mathfrak{h} \oplus C$, but only for C pointed and generating.

PROPOSITION 2.4. *For a wedge* $W = \mathfrak{h} \oplus C$ *in a symmetric Lie algebra* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *the following conditions are equivalent*:

(1) $T_x(W)$ *is a subalgebra for all* $x \in W$.

(2) W *is an Ol'shanskiĭ semialgebra.*

(3) $T_c(C)$ *is an* $\mathfrak{h}$*-module for all* $c \in C$.

If the wedge C *is generating, these conditions are equivalent to*

(3′) $T_c(C)$ *is an* $\mathfrak{h}$*-module for all* $c \in C^1(C)$.

PROOF. We omit the proof of this proposition which is similar to that of Proposition 2.2. □

Let V be a finite dimensional $\mathfrak{h}$-module. The dual space $\hat{V}$ is also an $\mathfrak{h}$-module in a canonical way. The action of $h \in \mathfrak{h}$ is given by $\langle h \cdot \omega, x \rangle = -\langle \omega, h \cdot x \rangle$ for $x \in V$, $\omega \in \hat{V}$. Assume T is a hyperplane in V which also is a submodule. Then we find a linear form $\alpha_T \in \hat{\mathfrak{h}}$ such that the action of $\mathfrak{h}$ on the factor space V/T is given by $h \cdot \xi = \alpha_T(h)\xi$. Since T is a hyperplane, the annihilator $T^{\perp} \overset{\text{def}}{=} \{\omega \in \hat{V} \mid \langle \omega, T \rangle = \{0\}\}$ is a one-dimensional submodule in $\hat{V}$, that is $T^{\perp} = \mathbb{R} \cdot \omega_T$. Then $h \cdot \omega_T = -\alpha_T(h)\omega_T$.

We apply this to the case where the tangent space $T_c(C)$ of a generating wedge $C \subseteq V$ is a hyperplane $\mathfrak{h}$-module. We denote the assigned linear form by $\omega_c = \omega_{T_c(C)} \in \hat{V}$ and $\alpha_c = \alpha_{T_c(C)} \in \hat{\mathfrak{h}}$.

LEMMA 2.5. *Let V be a finite dimensional $\mathfrak{h}$-module and C a pointed generating cone in V such that $T_c(C)$ is a submodule for every $c \in C^1(C)$. Then the following holds*:

(i) *V admits the decomposition*

$$V = \sum_{\alpha \in \Omega} V^{\alpha},$$

where $\Omega \subseteq \hat{\mathfrak{h}}$, and the weight spaces are given by $V^{\alpha} = \{x \in V \mid h \cdot x = \alpha(h)x \text{ for } h \in \mathfrak{h}\}$.

(ii) *We have $V^{\alpha} = \bigcap \{T_c(C) \mid c \in C^1(C), \alpha_c \neq \alpha\}$.*

(iii) *The wedges $\tilde{C}_{\alpha} \overset{\text{def}}{=} \bigcap \{L_c(C) \mid c \in C^1(C), \alpha_c = \alpha\}$ are generating wedges with edge $H(\tilde{C}^{\alpha}) = \sum_{\beta \neq \alpha} V^{\beta}$.*

(iv) *There are pointed generating cones $C_{\alpha} \subseteq V^{\alpha}$ such that $C = \sum_{\alpha \in \Omega} C_{\alpha}$.*

PROOF. (i) We consider the dual module $\hat{V}$. From [2] I.3.12 we conclude that $\{0\} = H(C) = \bigcap_{c \in C^1(C)} T_c(C)$. Thus we have $\hat{V} = \{0\}^{\perp} = (\bigcap_{c \in C^1(C)} T_c(C))^{\perp} = \sum_{c \in C^1(C)} \mathbb{R} \cdot \omega_c$, where $h \cdot \omega_c = -\alpha_c(h)\omega_c$ for $h \in \mathfrak{h}$. Thus $\hat{V}$ is completely reducible, and so is V.

Set $\Omega \overset{\text{def}}{=} \{\alpha_c\}$. For $\alpha \in \hat{\mathfrak{h}}$, we define $\hat{V}^{-\alpha} \overset{\text{def}}{=} \operatorname{span}\{\omega_c \mid \alpha_c = \alpha\}$. Then the root decomposition $\hat{V} = \sum_{\alpha \in \Omega} \hat{V}^{-\alpha}$ holds. With help of the annihilator mechanism, we get the analogous decomposition for V. In fact, with $V^{\alpha} \overset{\text{def}}{=} (\sum_{\beta \in \Omega \setminus \{\alpha\}} \hat{V}^{-\beta})^{\perp} \cong (\hat{V}^{-\alpha})\hat{}$ we have

$$V = \sum_{\alpha \in \Omega} V^{\alpha}.$$

(ii) We have $V_{\alpha} = (\sum_{\beta \in \Omega \setminus \{\alpha\}} \hat{V}^{\beta})^{\perp} = (\operatorname{span}\{\omega_c \mid \alpha_c \neq \alpha\})^{\perp} = \bigcap \{\omega_c^{\perp} \mid \alpha_c \neq \alpha\} = \bigcap \{T_c(C) \mid c \in C^1(C), \alpha_c \neq \alpha\}$. This shows (ii).

(iii) Since $C \subseteq \tilde{C}_{\alpha}$, the wedges $\tilde{C}_{\alpha}$ are obviously generating. To prove $H(\tilde{C}_{\alpha}) = \sum_{\beta \in \Omega \setminus \{\alpha\}} V^{\beta}$ we show the equivalent equation for the annihilators. We have

$$\begin{aligned} H(\tilde{C}_{\alpha})^{\perp} &= (\bigcap \{T_c(C) \mid c \in C^1(C), \omega_c \in \hat{V}^{-\alpha}\})^{\perp} \\ &= \operatorname{span}\{\mathbb{R} \cdot \omega_c \mid c \in C^1(C), \omega_c \in \hat{V}^{-\alpha}\} \\ &= \hat{V}^{-\alpha}. \end{aligned}$$

Since $(\hat{V}^{-\alpha})^{\perp} = \sum_{\beta \in \Omega \setminus \{\alpha\}} V^{\beta}$ this proves (iii).

(iv) Set $C_\alpha \overset{\text{def}}{=} \tilde{C}_\alpha \cap V^\alpha$. Then $\tilde{C}_\alpha = C_\alpha \oplus \sum_{\beta \in \Omega \setminus \{\alpha\}} V^\beta$. Therefore we have

$$C = \bigcap_{\alpha \in \Omega} \tilde{C}_\alpha = \sum_{\alpha \in \Omega} C_\alpha.$$

This proves the assertion. □

THEOREM 2.6. (Decomposition Theorem for Ol'shanskiĭ Semialgebras) *Let $W = \mathfrak{h} \oplus C$ be an Ol'shanskiĭ semialgebra in a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and assume C is pointed and generating. Then the $\mathfrak{h}$-module $\mathfrak{q}$ has a decomposition*

$$\mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}^+ \quad \textit{with} \quad \mathfrak{g}^+ = \sum_{\alpha \in \Omega} \mathfrak{q}^\alpha,$$

where $\Omega \subseteq \hat{\mathfrak{h}} \setminus \{0\}$. Further, the following holds:

(i) *The weight spaces are given by*

$$\mathfrak{q}_0 = \{x \in \mathfrak{q} \mid [h, x] = 0 \text{ for } h \in \mathfrak{h}\}$$
$$= \bigcap \{T_c(C) \mid c \in C^1(C), \alpha_c \neq 0\}.$$
$$\mathfrak{q}^\alpha = \{x \in \mathfrak{q} \mid [h, x] = \alpha(h)x \text{ for } h \in \mathfrak{h}\}$$
$$= \bigcap \{T_c(C) \mid c \in C^1(C), \alpha_c \neq \alpha\}.$$

(ii) *W is the intersection of halfspace Ol'shanskiĭ semialgebras and an invariant Ol'shanskiĭ wedge. More precisely,*

$$W = W_0 \cap \bigcap_{\alpha \in \Omega} W_\alpha,$$

where $W_0 = \mathfrak{h} \oplus \tilde{C}_0$ is invariant and $W_\alpha = \mathfrak{h} \oplus \tilde{C}_\alpha$ are semialgebras.

(iii) *The wedge W is adapted to the root decomposition in* (i). *There exist pointed generating cones C_0 in $\mathfrak{q}_0$ and C_α in $\mathfrak{q}_\alpha$ such that*

$$W = \mathfrak{h} \oplus C_0 \oplus \sum_{\alpha \in \Omega} C_\alpha.$$

PROOF. Most of the work is already done in Lemma 2.5. We remark, that in Proposition 2.6 the case $\alpha = 0$ is separated and Ω is supposed to be in $\hat{\mathfrak{h}} \setminus \{0\}$. In (i) and (iii) there is nothing left to show.

(ii) We must prove the fact that W_0 is invariant and W_α are semialgebras. Now, $W_0 = \mathfrak{h} \oplus \tilde{C}_0$ with $\tilde{C}_0$ from Lemma 2.5. Then $H(\tilde{C}_0) = \mathfrak{q}^+$. In particular $[\mathfrak{h}, \mathfrak{q}] \subseteq H(\tilde{C}_0)$. From Proposition 2.2 (3) we conclude that W_0 is an invariant Ol'shanskiĭ wedge.

Since $T_c(C)$ is a hyperplane $\mathfrak{h}$-module for every $c \in C^1(C)$, the wedge $W_c \overset{\text{def}}{=} \mathfrak{h} \oplus L_c(W)$ is a halfspace Ol'shanskiĭ semialgebra due to Proposition 2.4. But then $W_\alpha = \bigcap \{W_c \mid \alpha_c = \alpha\}$ is the intersection of halfspace Ol'shanskiĭ semialgebras. This proves (ii). □

We denote by $\mathfrak{m}(\mathfrak{h})$ the span of all one-dimensional ideals of a Lie algebra $\mathfrak{h}$. Then $\mathfrak{m}(\mathfrak{h})$ is an abelian ideal, called the *base ideal* of $\mathfrak{h}$. For each Lie algebra homomorphism $\varrho : \mathfrak{h} \to \mathbb{R}$ we define $\mathfrak{m}_\varrho(\mathfrak{h}) = \{y \in \mathfrak{h} \mid [x, y] = \varrho(x) y \text{ for all } x \in \mathfrak{h}\}$.

We call ϱ a *base root* if $\mathfrak{m}_\varrho(\mathfrak{h}) \neq \{0\}$. The set of all base roots is denoted by $B(\mathfrak{h})$, and $\mathfrak{m}_\varrho(\mathfrak{h})$ is called the *base root space of* ϱ. Each base root space $\mathfrak{m}_\varrho(\mathfrak{h})$ is a characteristic ideal and $\mathfrak{m}(\mathfrak{h})$ is the direct sum of the base root spaces.

THEOREM 2.7. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra supporting an Ol'shanskiĭ semialgebra $W = \mathfrak{h} \oplus C$ with C pointed and generating. Then the following holds*:

(i) *Let $\alpha, \beta \in \Omega \cup \{0\}$. Then* $[\mathfrak{q}^\alpha, \mathfrak{q}^\beta] \subseteq \mathfrak{m}_{\alpha+\beta}(\mathfrak{h})$.

(ii) $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{m}(\mathfrak{h})$. *In particular,* $[\mathfrak{q}, \mathfrak{q}]$ *is an abelian ideal of* $\mathfrak{h}$.

(iii) *If $\alpha, \beta, \gamma \in \Omega \cup \{0\}$ do not satisfy $\alpha = -\beta = \pm\gamma$, then* $[[\mathfrak{q}^\alpha, \mathfrak{q}^\beta], \mathfrak{q}^\gamma] = \{0\}$.

(iv) *If $\alpha, -\alpha \in \Omega \cup \{0\}$ and* $\dim \mathfrak{q}^\alpha > 1$, *then* $[[\mathfrak{q}^\alpha, \mathfrak{q}^{-\alpha}], \mathfrak{q}^\alpha] = \{0\}$.

(v) $[\mathfrak{q}_0, \mathfrak{q}]$ *is an abelian ideal of* $\mathfrak{g}$.

(vi) $[\mathfrak{q}_0, \mathfrak{q}_0] \subseteq \mathfrak{z}(\mathfrak{h})$, *and if $\alpha, -\alpha \in \Omega$ then* $[\mathfrak{q}^\alpha, \mathfrak{q}^{-\alpha}] \subseteq \mathfrak{z}(\mathfrak{h})$.

PROOF. (i) Let $x_\alpha \in \mathfrak{q}^\alpha$, $x_\beta \in \mathfrak{q}^\beta$ and $h \in \mathfrak{h}$. Then $[h, [x_\alpha, x_\beta]] = [[h, x_\alpha], x_\beta] + [x_\alpha, [h, x_\beta]] = (\alpha(h) + \beta(h))[x_\alpha, x_\beta]$. Therefore, if $[x_\alpha, x_\beta] \neq 0$, then $\alpha + \beta \in B(\mathfrak{h})$. This shows (i).

(ii) For the proof of (ii), we first notice that $[\mathfrak{q}, \mathfrak{q}]$ is an ideal of $\mathfrak{h}$. In fact, we have $[[\mathfrak{q}, \mathfrak{q}], \mathfrak{h}] \subseteq [\mathfrak{q}, \mathfrak{q}]$ by Jacobi identity. Since $[\mathfrak{q}, \mathfrak{q}] \subseteq \sum_{\alpha, \beta \in \Omega \cup \{0\}} [q^\alpha, \mathfrak{q}^\beta]$, the inclusion $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{m}(\mathfrak{h})$ follows from (i). But then $[\mathfrak{q}, \mathfrak{q}]$ is abelian, since $\mathfrak{m}(\mathfrak{h})$ is abelian.

(iii) Let $\alpha, \beta \in \Omega \cup \{0\}$. Then $[[\mathfrak{q}^\alpha, \mathfrak{q}^\beta], \mathfrak{q}^\gamma] \subseteq \mathfrak{q}^\gamma$. On the other hand the Jacobi identity shows that $[[\mathfrak{q}^\alpha, \mathfrak{q}^\beta], \mathfrak{q}^\gamma] \subseteq \mathfrak{q}^\alpha + \mathfrak{q}^\beta$ and $[[\mathfrak{q}^\alpha, \mathfrak{q}^\beta], \mathfrak{q}^\gamma] \subseteq \mathfrak{q}^{\alpha+\beta+\gamma}$. Since the decomposition $\mathfrak{q} = \mathfrak{q}_0 \oplus \sum_{\alpha \in \Omega} \mathfrak{q}^\alpha$ is direct, $[[\mathfrak{q}^\alpha, \mathfrak{q}^\beta], \mathfrak{q}^\gamma] \neq \{0\}$ implies $\alpha = -\beta = \pm\gamma$. This proves (iii).

(iv) If $\alpha = \{0\}$, we have nothing to show. Now let $\alpha, -\alpha \in \Omega$, and $x_\alpha, x'_\alpha \in \mathfrak{q}^\alpha$ and $x''_{-\alpha} \in \mathfrak{q}^{-\alpha}$ arbitrary elements. Then (iii) and the Jacobi identity show $\alpha([x'_\alpha, x''_{-\alpha}])x_\alpha - \alpha([x_\alpha, x''_{-\alpha}])x'_\alpha = 0$. If $\dim \mathfrak{q}^\alpha > 0$, take x_α and x'_α linearly independent. This shows $\alpha(x_\alpha, x''_{-\alpha}) = 0$. Since x_α and $x''_{-\alpha}$ are arbitrary, (iv) follows.

(v) From (iii) we conclude $[[\mathfrak{q}_0, \mathfrak{q}], \mathfrak{q}] = \{0\}$. Also, the inclusion $[[\mathfrak{q}_0, \mathfrak{q}], \mathfrak{h}] \subseteq [q_0, [\mathfrak{q}, \mathfrak{h}]] + [[\mathfrak{q}_0, \mathfrak{h}], \mathfrak{q}] = \{0\}$ holds. This shows (v).

(vi) We notice, that if $0 \in B(\mathfrak{h})$ then $\mathfrak{m}_0(\mathfrak{h}) = \mathfrak{z}(\mathfrak{h})$. Thus (vi) follows from (i). □

With these preparations, we can classify those symmetric Lie algebras which support a generating Ol'shanskiĭ semialgebra $W = \mathfrak{h} \oplus C$ with C pointed.

THEOREM 2.8. (Classification of symmetric Lie algebras with Ol'shanskiĭ semialgebras) *Let $\mathfrak{h}$ be a Lie algebra with base ideal $\mathfrak{m}(\mathfrak{h})$ and Ω a finite subset of $\hat{\mathfrak{h}}$ such that $\alpha \in \Omega$ induces a linear form on $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$. Let $\mathfrak{q} = \mathfrak{q}_0 \oplus \sum_{\alpha \in \Omega} \mathfrak{q}^\alpha$ be a direct sum of vector spaces and $\kappa : \mathfrak{q} \times \mathfrak{q} \to \mathfrak{m}(\mathfrak{h})$ a skew-symmetric bilinear map such that*

($\mathrm{J}_\mathfrak{h}$) *for* $\alpha, \beta \in \Omega \cup \{0\}$

$$\kappa(\mathfrak{q}^\alpha, \mathfrak{q}^\beta) \subseteq \mathfrak{m}_{\alpha+\beta}(\mathfrak{h}).$$

($J'_{\mathfrak{q}}$) *for all* $\alpha, \beta, \gamma \in \Omega \cup \{0\}$

$$\kappa(\mathfrak{q}^\alpha, \mathfrak{q}^\beta) \subseteq \ker \gamma \quad \textit{unless } \alpha = -\beta = \pm\gamma.$$

($J''_{\mathfrak{q}}$) *if* $\dim \mathfrak{q}^\alpha > 1$, *then*

$$\kappa(\mathfrak{q}^\alpha, \mathfrak{q}^{-\alpha}) \subseteq \ker \alpha.$$

Then the set $\mathfrak{g}_{\kappa,\Omega} \overset{\text{def}}{=} \mathfrak{h} \oplus \mathfrak{q}$ *with the bracket*

$$[(h,q),(h',q')] = ([h,h'] + \kappa(q,q'), \sum_{\alpha \in \Omega} (\alpha(h)x'_\alpha - \alpha(h')x))$$

(*for* $q = x_0 + \sum x_\alpha$, $q' = x'_0 + \sum x'_\alpha \in \mathfrak{q}$, $h, h' \in \mathfrak{h}$) *is a Lie algebra.*

If $C_0 \subseteq \mathfrak{q}_0$, $C_\alpha \subseteq \mathfrak{q}^\alpha$, $\alpha \in \Omega$, *are pointed generating cones, then* $W = \mathfrak{h} \oplus C$ *with* $C = C_0 \oplus \sum_{\alpha \in \Omega} C_\alpha$ *is an Ol'shanskiĭ semialgebra.*

Conversely, if the symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *supports an Ol'shanskiĭ semialgebra* $W = \mathfrak{h} \oplus C$ *with C pointed and generating, then* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{g}_{\kappa,\Omega}$ *where* $\kappa(q,q') = [q,q']$ *for* $q, q' \in \mathfrak{q}$ *and* $\mathfrak{q} = \mathfrak{q}_0 \oplus \sum_{\alpha \in \Omega} \mathfrak{q}^\alpha$ *is the root decomposition of Theorem* 2.6.

PROOF. We first show that $\mathfrak{g}_{\kappa,\Omega}$ is a Lie algebra. The skew-symmetry and bilinearity of $[\cdot,\cdot]$ are obvious. Let $q, q', q'' \in \mathfrak{q}$ and $h, h', h'' \in \mathfrak{h}$. Then

$$\begin{aligned}
&[(h,q),[(h',q'),(h'',q'')]] \\
&\quad = ([h,[h',h'']] + [h,\kappa(q',q'')] + \kappa(q, \sum_{\alpha \in \Omega} (\alpha(h')x''_\alpha - \alpha(h'')x'_\alpha)), \\
&\qquad \sum_{\alpha \in \Omega} (\alpha(h)\alpha(h')x''_\alpha - \alpha(h)\alpha(h'')x'_\alpha) - \sum_{\alpha \in \Omega} \alpha(\kappa(q',q''))x_\alpha),
\end{aligned}$$

where we used $[\mathfrak{h},\mathfrak{h}] \subseteq \ker \alpha$. We first examine the second component. By ($J'_{\mathfrak{q}}$) the second sum reduces to $\sum_{\alpha \in \Omega} \alpha(\kappa(x'_\alpha, x''_{-\alpha}))x_\alpha + \alpha(\kappa(x'_{-\alpha}, x''_\alpha))x_\alpha$. We add the terms which we get after cyclic permutation of the argument. If $\dim \mathfrak{q}^\alpha = 1$ the corresponding sum vanishes, since κ is bilinear and skew-symmetric. By ($J''_{\mathfrak{q}}$), the term also vanishes in the case $\dim \mathfrak{q}^\alpha > 1$. Cyclic permutation of the arguments in the first sum shows that the Jacobi identity holds for the second component.

In the same manner, ($J_{\mathfrak{h}}$) implies the Jacobi identity in the first component. This shows that $\mathfrak{g}_{\kappa,\Omega}$ is a Lie algebra.

Now, let $C_0 \subseteq \mathfrak{q}_0$, $C_\alpha \subseteq \mathfrak{q}^\alpha$, $\alpha \in \Omega$, be any pointed generating cones and $W = \mathfrak{h} \oplus C$ with $C = C_0 \oplus \sum_{\alpha \in \Omega} C_\alpha$. Set $W_0 = \mathfrak{h} \oplus C_0 \oplus \mathfrak{q}^+$ and $W_\alpha = \mathfrak{h} \oplus \mathfrak{q}_0 \oplus \sum_{\beta \in \Omega \setminus \{\alpha\}} \mathfrak{q}^\beta \oplus C_\alpha$. Then $W = W_0 \cap \bigcap_{\alpha \in \Omega} W_\alpha$.

Claim: W_α is a semialgebra and W_0 is an invariant wedge. All wedges are generating. First, let $x = h + x_\alpha + y \in C^1(W_\alpha)$ with $h \in \mathfrak{h}$, $x_\alpha \in \mathfrak{q}^\alpha$ and $y \in \mathfrak{q}_0 \oplus \sum_{\beta \in \Omega \setminus \{\alpha\}} \mathfrak{q}^\beta$. We have $x \in C^1(W_\alpha)$ if and only if $x_\alpha \in C^1(C_\alpha)$, and $T_x(W_\alpha) = \mathfrak{h} \oplus \sum_{\beta \in \Omega \setminus \{\alpha\}} \mathfrak{q}^\beta \oplus T_{x_\alpha}(C_\alpha)$. We have to show $[T_x(W_\alpha), x] \subseteq T_x(W_\alpha)$. By $[\mathfrak{h}, x_\alpha] \subseteq \alpha(\mathfrak{h})x_\alpha \subseteq T_{x_\alpha}(C_\alpha)$ and $[T_{x_\alpha}(C_\alpha), x_{\mathfrak{h}}] \subseteq T_{x_\alpha}(C_\alpha)$, this is straightforward. In the same way $[\mathfrak{g}, x] \subseteq T_x(W_0)$ for all $x \in W_0$ follows. This proves the claim.

But then, W is an Ol'shanskiĭ semialgebra as an intersection of an invariant wedge and semialgebras.

Conversely, let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ be a symmetric Lie algebra supporting an Ol'shanskiĭ semialgebra $W=\mathfrak{h}\oplus C$ with C pointed and generating. Define κ by $\kappa(q,q')=[q,q']$ for $q,q'\in\mathfrak{q}$. Then Theorem 2.6 and Theorem 2.7 prove the assertion. □

REMARK 2.9. Condition $(\mathrm{J}_{\mathfrak{h}})$ describes the fact, that $\kappa(\mathfrak{q},\mathfrak{q})\subseteq\mathfrak{m}(\mathfrak{h})$. Thus if $\alpha\in\Omega$ is also a base root, then $\kappa(\mathfrak{q},\mathfrak{q})\subseteq\ker\alpha$ is immediate since $\mathfrak{m}(\mathfrak{h})$ is abelian, so, $\mathfrak{m}(\mathfrak{h})\subseteq\ker\varrho$ for every $\varrho\in B(\mathfrak{h})$.

From [5] we recall the definition of the *Δ-radical* $\Delta(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. It is the intersection of all hyperplane subalgebras and is a fully characteristic ideal. The intersection $\Delta_s(\mathfrak{g})$ of all hyperplane subalgebras of the simple type, called the *s-radical*, and the intersection $\Delta_a(\mathfrak{g})$ of hyperplane subalgebras of the abelian or solvable type, the *a-radical*, are also fully characteristic ideals with $\Delta_s(\mathfrak{g})\cap\Delta_a(\mathfrak{g})=\Delta(\mathfrak{g})$. For definitions and further details we refer to [4], [5].

We notice that the second term $\mathfrak{g}''$ of the commutator series of $\mathfrak{g}$ is contained in $\Delta_a(\mathfrak{g})$, and that for the radical $\mathfrak{r}$ the inclusion $\mathfrak{r}\subseteq\Delta_s(\mathfrak{g})$ holds. If $\mathfrak{s}$ is any Levi complement then we have $\mathfrak{s}\subseteq\mathfrak{g}''\subseteq\Delta_a(\mathfrak{g})$. If $\mathfrak{g}$ is *Δ-reduced*, which means that $\Delta(\mathfrak{g})=\{0\}$, then $\mathfrak{g}=\Delta_a(\mathfrak{g})\oplus\Delta_s(\mathfrak{g})$, where $\mathfrak{r}=\Delta_s(\mathfrak{g})$ and $\mathfrak{s}=\Delta_a(\mathfrak{g})\cong\mathrm{sl}(2,\mathbb{R})^m$ for a suitable m is the unique Levi complement.

LEMMA 2.10. *Let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ be a symmetric Lie algebra with radical $\mathfrak{r}$. Then we have $\mathfrak{r}=\mathfrak{r}_{\mathfrak{h}}\oplus\mathfrak{r}_{\mathfrak{q}}$, $\Delta(\mathfrak{g})=\Delta(\mathfrak{g})_{\mathfrak{h}}\oplus\Delta(\mathfrak{g})_{\mathfrak{q}}$, $\Delta_s(\mathfrak{g})=\Delta_s(\mathfrak{g})_{\mathfrak{h}}\oplus\Delta_s(\mathfrak{g})_{\mathfrak{q}}$, and $\Delta_a(\mathfrak{g})=\Delta_a(\mathfrak{g})_{\mathfrak{h}}\oplus\Delta_a(\mathfrak{g})_{\mathfrak{q}}$, where the subscripts denote the intersection with $\mathfrak{h}$, resp., $\mathfrak{q}$.*

PROOF. Let $\tau:\mathfrak{g}\to\mathfrak{g}$ denote the involutive Lie algebra automorphism corresponding to $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$. We have to show that the indicated ideals are τ-invariant. Firstly, $\tau(\mathfrak{r})$ also is a solvable ideal, hence $\tau(\mathfrak{r})=\mathfrak{r}$. This proves the assertion concerning $\mathfrak{r}$. If T is a hyperplane subalgebra, then $\tau(T)$ also is a hyperplane subalgebra of the same type. Hence $\Delta(\mathfrak{g})$, $\Delta_s(\mathfrak{g})$, and $\Delta_a(\mathfrak{g})$ are τ-invariant. □

LEMMA 2.11. *If the symmetric Lie algebra $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ supports a generating Ol'shanskiĭ semialgebra $W=\mathfrak{h}\oplus C$, then $\Delta(\mathfrak{g})$ is contained in the edge $H(W)$. If C is pointed, then $\Delta(\mathfrak{g})\subseteq\mathfrak{h}$.*

PROOF. We have $H(W)=\bigcap_{x\in C^1(W)}T_x(W)$ and all $T_x(W)$ are hyperplane subalgebras by Proposition 2.4. But then $\Delta(\mathfrak{g})\subseteq H(W)$ is obvious. If C is pointed, then $H(W)=\mathfrak{h}$. This proves the lemma. □

A generating Lie semialgebra W is called *reduced*, if the edge $H(W)$ contains no non-trivial ideal of $\mathfrak{g}$. With the preceding remarks, we can describe the Lie algebras which contain reduced Ol'shanskiĭ semialgebras.

THEOREM 2.12. *If the symmetric Lie algebra $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ supports a reduced Ol'shanskiĭ semialgebra $W=\mathfrak{h}\oplus C$ with C pointed, then $\mathfrak{g}=\mathrm{sl}(2,\mathbb{R})^m\oplus\mathfrak{r}$ with*

$\mathfrak{r}''=\{0\}$ *and* $[\mathfrak{r}_\mathfrak{q},\mathfrak{r}_\mathfrak{q}]=\{0\}$ *where* $\mathfrak{r}_\mathfrak{q}=\mathfrak{r}\cap\mathfrak{q}$. *There exist generating Ol'shanskiĭ semialgebras* $W_\mathfrak{s}\subseteq \mathrm{sl}(2,\mathbb{R})^m$ *and* $W_\mathfrak{r}\subseteq\mathfrak{r}$ *such that* $W=W_\mathfrak{s}\oplus W_\mathfrak{r}$.

PROOF. By Lemma 2.11, the ideal $\Delta(\mathfrak{g})$ is contained in $H(W)$. Thus $\Delta(\mathfrak{g})=\{0\}$, since W is reduced. Thus $\mathfrak{g}=\mathrm{sl}(2,\mathbb{R})^m\oplus\mathfrak{r}$ by what is said above. Also, $\mathfrak{s}\subseteq\mathfrak{g}''\subseteq\Delta_a(\mathfrak{g})=\mathfrak{s}$, in particular $\mathfrak{r}''=\{0\}$. By Lemma 2.10, we have $\mathfrak{g}=\mathfrak{s}_\mathfrak{h}\oplus\mathfrak{s}_\mathfrak{q}\oplus\mathfrak{r}_\mathfrak{h}\oplus\mathfrak{r}_\mathfrak{q}$. The $\mathfrak{h}$-module $\mathfrak{r}_\mathfrak{q}$ inherits the decomposition of the $\mathfrak{h}$-module $\mathfrak{q}$, and $(\mathfrak{r}_\mathfrak{q})^\alpha=\mathfrak{r}_\mathfrak{q}\cap\mathfrak{q}^\alpha$. To prove that $[\mathfrak{r}_\mathfrak{q},\mathfrak{r}_\mathfrak{q}]=\{0\}$ holds, it is enough to show that $[[\mathfrak{r}_\mathfrak{q},\mathfrak{r}_\mathfrak{q}],\mathfrak{q}]=\{0\}$. Indeed, if this holds, $[\mathfrak{r}_\mathfrak{q},\mathfrak{r}_\mathfrak{q}]$ is an ideal of $\mathfrak{g}$ contained in W, hence trivial, as W is reduced. Assume that $[[\mathfrak{r}_\mathfrak{q},\mathfrak{r}_\mathfrak{q}],\mathfrak{q}]\neq\{0\}$. Then there is an $\alpha\in\Omega$ and an $h\in[\mathfrak{r}_\mathfrak{q},\mathfrak{r}_\mathfrak{q}]$ with $\alpha(h)\neq 0$. Let $0\neq x\in\mathfrak{q}^\alpha\cap\mathfrak{r}$. Then $[h,x]=\alpha(h)x\neq 0$. But then $[h,x]=1/(\alpha(h))[h,[h,x]]\in\mathfrak{r}''=\{0\}$, which gives a contradiction. This shows $[\mathfrak{r}_\mathfrak{q},\mathfrak{r}_\mathfrak{q}]=\{0\}$.

Now let W be a reduced Ol'shanskiĭ semialgebra and $w\in C^1(W)$. Then $T_w(W)$ is a hyperplane subalgebra. If $T_w(W)$ is of the abelian or the solvable type, then $\mathfrak{s}=\Delta_a(\mathfrak{g})\subseteq T_w(W)$, and if it is of the semisimple type, then $\mathfrak{r}=\Delta_s(\mathfrak{g})\subseteq T_w(W)$. Set $\tilde{W}_\mathfrak{s}\overset{\mathrm{def}}{=}\bigcap\{L_w(W)\mid T_w(W)$ is of solvable or abelian type$\}$. Then $\tilde{W}_\mathfrak{s}$ is a generating Ol'shanskiĭ semialgebra of $\mathfrak{g}$ with $\mathfrak{r}\subseteq\tilde{W}_\mathfrak{s}$. Similarly, $\tilde{W}_\mathfrak{r}\overset{\mathrm{def}}{=}\bigcap\{L_w(W)\mid T_w(W)$ is of semisimple type$\}$ is a generating Ol'shanskiĭ semialgebra with $\mathfrak{s}\subseteq\tilde{W}_\mathfrak{r}$. Obviously, we have $W=\tilde{W}_\mathfrak{s}\cap\tilde{W}_\mathfrak{r}$. Define $W_\mathfrak{s}\overset{\mathrm{def}}{=}\tilde{W}_\mathfrak{s}\cap\mathfrak{s}$ and $W_\mathfrak{r}\overset{\mathrm{def}}{=}\tilde{W}_\mathfrak{r}\cap\mathfrak{r}$. Since $\mathfrak{s}\subseteq\tilde{W}_\mathfrak{r}$ and $\mathfrak{r}\subseteq\tilde{W}_\mathfrak{s}$, we have $W=W_\mathfrak{s}\oplus W_\mathfrak{r}$, and $W_\mathfrak{s}$, resp., $W_\mathfrak{r}$ are generating Ol'shanskiĭ semialgebras in $\mathfrak{s}$, resp., $\mathfrak{r}$. This proves the theorem. □

We remark, that the generating Lie semialgebras in $\mathrm{sl}(2,\mathbb{R})^m$ are precisely of the form $W_1\oplus\cdots\oplus W_m$ with generating Lie semialgebras W_j in $\mathrm{sl}(2,\mathbb{R})$. Thus the Ol'shanskiĭ semialgebras $W_\mathfrak{s}$ are classified. In the case that $W_\mathfrak{r}$ is pointed, Theorem 2.8 gives the classification of all possible wedges $W_\mathfrak{r}$. Then, in Theorem 2.8, we have to choose $\mathfrak{h}$ with $\mathfrak{h}''=\{0\}$ and we have to set $\kappa\equiv 0$, thus conditions $(\mathrm{J}_\mathfrak{q})$ and $(\mathrm{J}_\mathfrak{h})$ vanish.

REFERENCES

1. Dörr, N. - On Ol'shanskiĭ's semigroup, Math. Ann. **288**, 21-33 (1990).
2. Hilgert, J., K.H. Hofmann and J.D. Lawson - Lie Groups, Convex Cones, and Semigroups, Oxford Univ. Press, 1989.
3. Hilgert, J., G. 'Olafsson and B. Ørsted - Hardy spaces of affine symmetric spaces, to appear in Journ. der reinen und angew. Math.
4. Hofmann, K.H. - Lie algebras with subalgebras of codimension one, Illinois J. Math. **9**, 636-643 (1965).
5. Hofmann, K.H. - Hyperplane subalgebras of real Lie algebras, Geom. Dedicata **36**, 207-224 (1990).
6. Ol'shanskiĭ, G.I. - Invariant cones in Lie algebras, Lie semigroups, and the holomorphic discrete series, Funct. Anal. Appl. **15**, 275-285 (1982).